Algorithms for Structural Natural-Frequency Design

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The algorithms used in the JPL-IDEAS antenna-structure-design-optimization program are furnished here. The algorithms are based upon the operational research method of optimality criteria and the structural analysis method of virtual work. Examples of the natural-frequency-constrained design of an antenna tripod structure are included.

I. Introduction

The JPL-IDEAS program [1] is a finite-element structure-design-optimization program with minimum structure weight as the design objective. In addition to the more conventional constraints, such as those on stresses and displacements, structures can be subject to constraints on antenna microwave performance parameters [2]. The program can also accept minimum structural natural frequency [3] for any specified vibration mode as a constraint. The design variables are the areas of rod members or the thicknesses of shear, triangular, or quadrilateral membrane plates. The design approach employs the Optimality Criteria Method [4] in which Lagrangian multipliers [5,6] are used to determine the sizes of the design variables, and a virtual work formulation is used to determine the sensitivities of the design variables.

Details of the natural-frequency design algorithm used in an earlier version of the JPL-IDEAS program are described in [3]. As explained there, depending upon details of formulation of the optimization problem, it was necessary to use the artifice of scaling the final design to obtain the minimum weight design that met the natural-frequency constraints. In other research that also concentrates on the natural-frequency design case [7,8,9], it is also necessary to resort to scaling or recursion to obtain the final design. More recently, the JPL-IDEAS natural-frequency design algorithm has been improved to eliminate the need for scaling the design. Scaling is especially objectionable if members are to be selected from tables of commercially available structural shapes. The current algorithm provides an explicit equation for computation of the Lagrangian multiplier in the case of a single-mode natural-frequency constraint, and it can be extended to deal with constraints on multiple-mode frequencies. This algorithm is described here.

II. Sensitivity Coefficient

To arrive at the sensitivity of natural frequency to the design variables, the natural-frequency eigenvalue of a particular vibration mode is expressed in terms of the Rayleigh quotient as

$$\omega^2 = \frac{\phi^t \mathbf{K} \phi}{\phi^t \mathbf{M} \phi} \tag{1}$$

in which ϕ is the mode-shape eigenvector, $\mathbf{K}$ is the assembled structure stiffness matrix, M is the assembled structure lumped (diagonal) mass matrix, ω is the natural circular frequency, and ω^2 is the eigenvalue. Taking the partial derivative of the eigenvalue with respect to a particular design variable a_i (such as a rod area or plate thickness, either as an individual member or linked member group) and also using the customary definitions of generalized modal mass $\mathcal{M}$ and generalized modal stiffness $\mathcal{K}$, e.g.,

$$\mathcal{M} = \phi^t \mathbf{M} \phi \tag{2}$$

$$\mathcal{K} = \phi^t \mathbf{K} \phi \tag{3}$$

the partial derivative can be expressed as

$$\frac{\partial \omega^2}{\partial a_i} = \frac{1}{\mathcal{M}} \left[\phi^t \frac{\partial \mathbf{K}}{\partial a_i} \phi - \phi^t \frac{\partial \mathbf{M}}{\partial a_i} \phi \omega^2 \right] \tag{4}$$

in which the partial derivatives of the assembled stiffness and mass matrices can be expressed in terms of element member stiffness k_i and mass m_i matrices for the ith member as

$$\frac{\partial \omega^2}{\partial a_i} = \frac{1}{\mathcal{M} a_i} \left[\phi^t k_i \phi - \phi^t m_i \phi \omega^2 \right] \tag{5}$$

In the right side of the above equation, the first term in the brackets represents the virtual work and the second term the virtual kinetic energy for this element in this vibration mode. Note that each of the bracketed terms involves only the small subset of the eigenvector associated with the connectivity of the element. The bracketed term is replaced by the symbol V_i, which will be considered as an expression of vibratory virtual work, as modified by a subtractive kinetic energy term, e.g.,

$$V_i = \left[\phi^t k_i \phi - \phi^t m_i \phi \omega^2 \right] \tag{6}$$

Consequently, the partial derivative of the eigenvalue with respect to the ith member design variable is given by

$$\frac{\partial \omega^2}{\partial a_i} = \frac{V_i}{\mathcal{M} a_i} \tag{7}$$

It is convenient to design for structural natural frequency as much as possible within the code that is already in place for static loading design. The JPL-IDEAS program uses the virtual work of each design variable to design for static loading. In contrast to Eq. (6), this virtual work is expressed by a displacement-method formulation for plate members and by a force-method formulation for bar members as follows:

$$V_i = \phi_R^t k_i \phi_D = \left(C_R C_D \frac{\ell}{aE} \right)_i \tag{8}$$

in which the subscript R denotes a real external loading and D denotes a virtual external loading, ϕ_R represents the corresponding real displacement vector, and ϕ_D represents the virtual loading displacement vector. For a bar element, C_R and C_D represent real loading and virtual loading stress resultants, while ℓ, a, and E represent the corresponding length, area, and modulus of elasticity. To simplify the exposition, the remaining discussion will emphasize the treatment for bar members. Minor modifications appropriate to treat plate members can be found in [3].

In the case of bar members, it can be seen that the virtual work for the natural-frequency design can have the identical form of that in Eq. (8) for the static-loading design when using the second equality of this equation, provided that C_R is interpreted as the stress resultant corresponding to the displacements of the eigenvector, and C_D (dropping the subscript i) is computed as

$$C_D = C_R - \omega^2 \phi^t m \phi \frac{aE}{\ell C_R} \tag{9}$$

Furthermore, if, as in [6], a combined stress coefficient term FIJ' is defined as

$$FIJ' = C_R \frac{C_D}{E} \tag{10}$$

the virtual work for the ith element (again dropping the subscript) can be expressed as

$$V_i = \left(\frac{FIJ'\ell}{a} \right)_i \tag{11}$$

Consequently, the sensitivity of the eigenvalue to the ith bar design variable is given by

$$\frac{\partial \omega^2}{\partial a_i} = \left(\frac{FIJ'\ell}{a^2} \right)_i \cdot \frac{1}{\mathcal{M}} \tag{12}$$

At this point, it is convenient to replace FIJ' with

$$FIJ = \frac{FIJ'}{\mathcal{M}} \tag{13}$$

so that the sensitivity equation becomes

$$\frac{\partial \omega^2}{\partial a_i} = \left(\frac{FIJ\ell}{a^2}\right)_i \tag{14}$$

III. Optimality Criterion

Let ω^* be the minimum requirement for the natural frequency; the following constraint inequality then applies:

$$\omega^{*2} - \omega^2 \leq 0 \tag{15}$$

The object is to minimize the structural weight, which, for bar members with a weight-density parameter γ, is given by

$$\text{objective} = \min\left(\sum(\gamma\ell a)_i\right) \tag{16}$$

Forming the Lagrangian $\mathbf{L}^*$ in the conventional way, with λ as the yet-to-be-determined Lagrangian multiplier, provides

$$\mathbf{L}^* = \sum(\gamma\ell a)_i + \lambda(\omega^{*2} - \omega^2) \tag{17}$$

Setting the partial derivative of the expression with respect to each design variable equal to zero and using Eq. (14) provides the explicit expression for the optimum value of the design variable $\bar{a}_i$ as follows:

$$\bar{a}_i = \left[\lambda\left(\frac{FIJ}{\gamma}\right)_i\right]^{1/2} \tag{18}$$

The above equation defines the optimality criterion for each of the design variables. The remaining requirement is to determine λ.

IV. Lagrangian Multiplier

Using the sensitivity expression and summing over all the members, $\Delta\omega^2$ (the change in eigenvalue) can be estimated as follows:

$$\Delta\omega^2 = \sum\left[\left(\frac{FIJ\ell}{a^2}\right)(\bar{a} - a)\right]_i \tag{19}$$

Writing the right side of the above equation as the difference of two summations, the following is obtained:

$$\Delta\omega^2 = \sum\left(\frac{FIJ\ell\bar{a}}{a^2}\right)_i - \sum\left(\frac{FIJ\ell}{a}\right)_i \tag{20}$$

By reexamining Eqs. (6), (8), and (10) and with some algebra, it can be shown that the second summation in Eq. (20) can be written as

$$\sum\left(\frac{FIJ\ell}{a}\right)_i = \omega^2\frac{\mathbf{M_F}}{\mathcal{M}} \tag{21}$$

in which $\mathbf{M_F}$ is the portion of the generalized mass matrix contributed by the nonstructural (parasitic) masses. That is, the generalized mass can be considered as the sum of the contributions from the structural-design variables $\mathbf{M_S}$ and the contribution of the fixed masses, $\mathbf{M_F}$; e.g.,

$$\mathcal{M} = \mathbf{M_S} + \mathbf{M_F} \tag{22}$$

Solving the constraint equation as an equality provides

$$\Delta\omega^2 = \omega^{*2} - \omega^2 \tag{23}$$

The first summation on the right side of Eq. (20) can be written as the sum of two terms; one term depends upon free design variables that can be determined according to the optimality criteria; the second term has bounds a_b for the design variables, such as a side constraint or a move limit. Therefore, this term can be written as

$$\sum\left(\frac{FIJ\ell\bar{a}}{a^2}\right)_i = \sum_{free}\left(\frac{FIJ\ell}{a^2}\left(\frac{FIJ}{\gamma}\right)^{1/2}\right)_i \lambda^{1/2}$$

$$+ \sum_{bounded}\left(\frac{FIJ\ell}{a^2}a_b\right)_i \tag{24}$$

Finally, some algebra based upon using Eqs. (20), (21), (22), (23), and (24) provides an explicit expression to evaluate the Lagrangian multiplier as follows:

$$\lambda^{1/2} = \frac{\left(\omega^{*2} - \omega^2 \dfrac{\mathbf{M}_s}{\mathcal{M}} - \displaystyle\sum_{bounded} \frac{FIJ\ell}{a^2} a_b\right)}{\displaystyle\sum_{free} \frac{FIJ\ell}{a^2} \left(\frac{FIJ}{\gamma}\right)^{1/2}} \qquad (25)$$

Consequently, Eq. (18) in conjunction with Eq. (25) constitutes the solution to the natural-frequency design problem for a particular natural mode. The foregoing developments could also be extended to provide the solution for simultaneous constraints on the frequencies of several modes. These equations are applied iteratively in a sequence of design cycles with move limits. Move limits compensate for linearizations inherent to the foregoing formulations.

V. Design Examples

A hypothetical test problem is chosen to demonstrate design optimization. The structure considered is a modification of the actual DSN DSS 13 antenna-reflector tripod structure. Figure 1 shows the antenna and tripod assembly just after completion in the summer of 1990. Figure 2 shows the layout and geometry of the isolated tripod structure component, which is designed with constraints on minimum natural frequency. The analytical model contains about 350 individual structural trusswork rod and plate members, which are linked into 40 distinct design variable groups. There are about 200 nodes and 450 unrestrained degrees of freedom. In these examples, members were selected from a continuous spectrum of available sizes. In practice, there is a program option that causes members to be selected from discrete tables of commercially available structural shapes.

The frequency constraints for the first three natural modes were postulated to be 2.5, 5.2, and 5.8 Hz. The starting design consisted of about 4.8 kilopounds (kips) of structure weight plus 4.5 kips of parasitic, nonstructure weight. All three frequencies at the start were each about 20 percent less than the constraint.

Figure 3(a) shows the frequencies achieved for independent designs for each of these modes. The designs of the second and third modes approximately satisfied the first mode constraint. The design for the second mode did not satisfy the third mode constraint, nor did the third mode design satisfy the second mode constraint. The first mode design did not satisfy either of the other two constraints. The structure weights achieved during these designs are shown in Fig. 3(b). It can be seen that both the second and third mode designs entailed substantial weight increases. Nevertheless, the first mode design resulted in a weight reduction while providing the desired increase in natural frequency.

In another example, all three constraints were applied simultaneously by using an envelope method as an approximation to a multiple-constraint design. This method treats the constraints sequentially and maintains the values of the design variables at a level no less than the values determined for previously treated constraints. Although not strictly an optimal procedure, it has often been found to work well in practice, especially for stress and displacement constraints. The history of the simultaneous designs and the structure weight is shown in Fig. 4. All the constraints are essentially satisfied at the ninth design cycle, and the structure weight is less than 3 percent greater than that of the isolated design for the third mode.

VI. Conclusion

The algorithm used to design for minimum natural frequency and in the JPL-IDEAS structural optimization computer program has been described and demonstrated by an example. The optimality criteria method, which is simple in concept and in execution, is employed. This method provides an explicit algorithm—almost trivial to invoke—to size the design variables. Determination of the Lagrangian multiplier, which is used in the algorithm, requires most of the computational effort. The algorithm is based on the well-known virtual work, dummy-load concept. The final design to meet the constraints is achieved directly and avoids the artifice of uniform scaling of computer-derived results.

Acknowledgment

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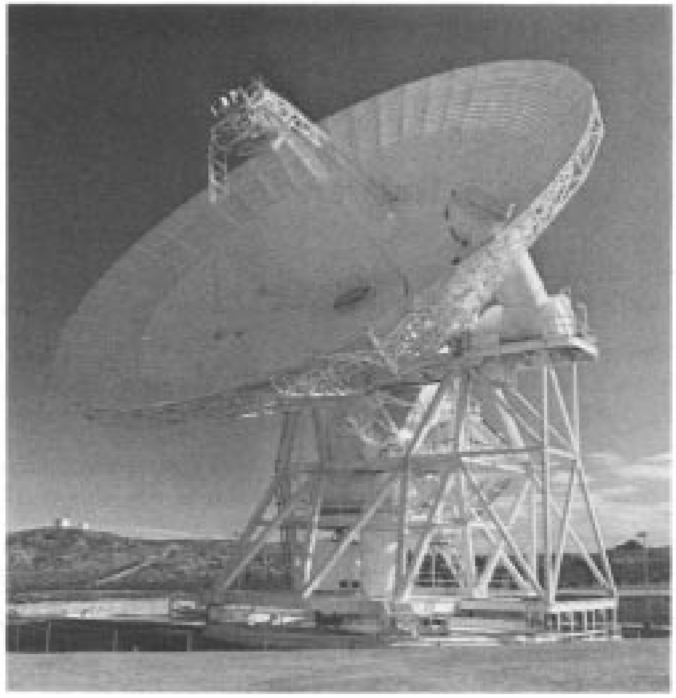

Fig. 1. DSN DSS-13 antenna-reflector-tripod structure.

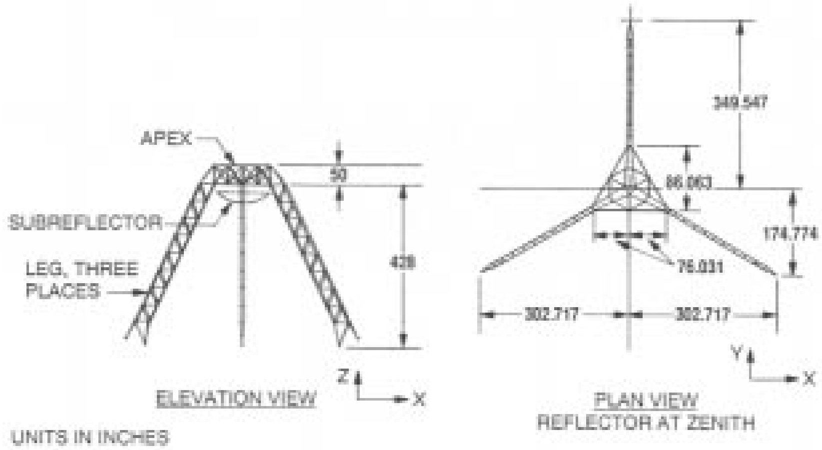

Fig. 2. DSS-13 tripod geometry.

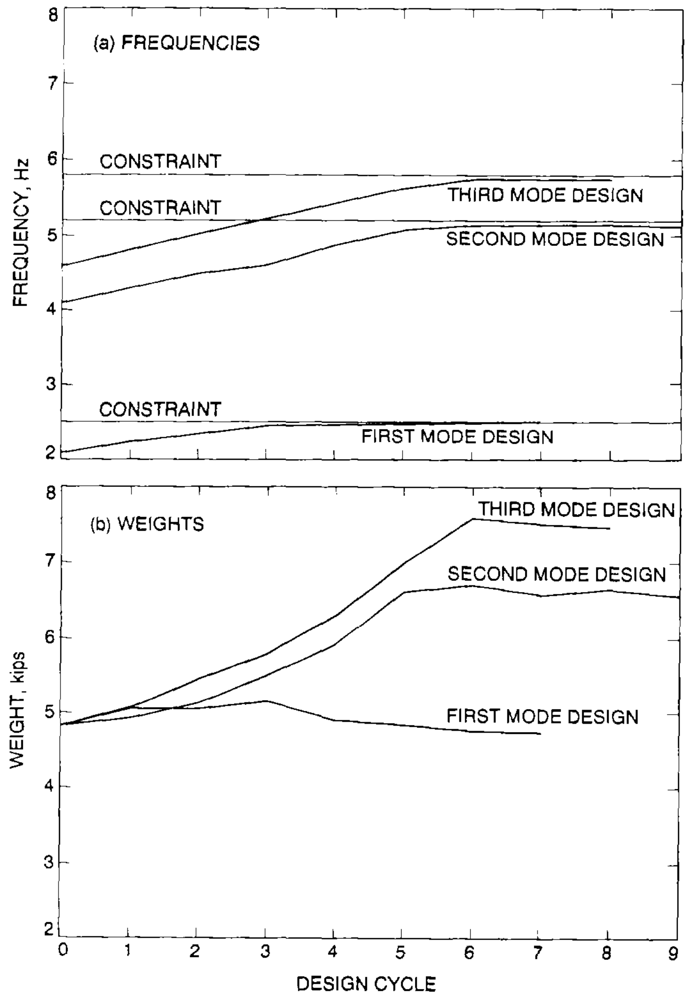

Fig. 3. Independent designs for single modes.

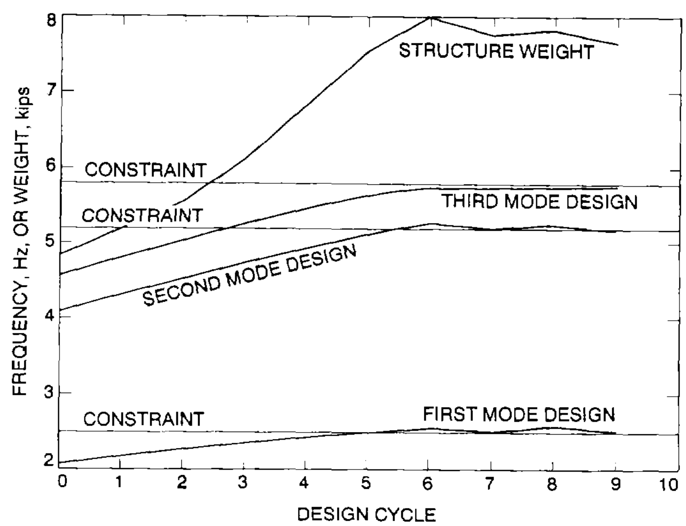

Fig. 4. Simultaneous constraints for three modes.